Sylvester's Criterion

Let A be a symmetric $n \times n$ real matrix.

Definition 1. *A is positive definite means*

$$X^T AX \geq 0,$$

for all column vectors X and

$$X^T AX = 0 \implies X = 0.$$

Definition 2. *A principal submatrix of a matrix is a matrix with row and column indices of the form $1, 2, \ldots, k$ (same sets of indices beginning with 1 and including all indices up to k). A principal minor is the determinant of a principal sub matrix. A special submatrix is a matrix with the same row and column indices (not necessarily starting with 1 and perhaps omitting some indices). A special minor is the determinant of a special submatrix.*

Theorem 1. *(Sylvester). A symmetric matrix A is positive definite if and only if all principal minors are positive. If A is positive definite all special minors are positive.*

Proof.

Lemma 1. *Let B be an invertible matrix. A is positive definite if and only if $B^T AB$ is positive definite.*

Proof. Suppose A is positive definite. Then $X^T B^T ABX = Y^T AY \geq 0$, where $Y = BX$. If $X^T B^T ABX = 0$, then $BX = 0$. Since B is invertible, $X = 0$. So A positive definite implies $B^T AB$ is positive definite. The reverse implication follows in the same way. □

Assume all principal minors of A are positive. Let $A = [a_{ij}]$. By assumption $a_{11} > 0$. Subtract multiplies of the first row of A to zero out the entries in the first column of A below the first. This operation preserves the values of the principal minors of A, so they remain positive. Next subtract multiples of the first column of the modified A matrix from the other columns of A to zero out the entries in the first row of A to the right of the first column. Principal minors are preserved. The first operation can be encoded by performing it to I (call the new matrix B) and multipling on the left. The second operation can be encoded by multiplying BA on the right by B^T. The principal minors of BAB^T are exactly the same as the original principal minors of A (and hence positive). There is a new $2, 2$ entry in BAB^T, but since it occurs in the lower right corner of 2×2 principal matrix with positive determinant and positive upper corner, it is positive and can be used to zero out entries in the second column below the second entry and then the entries in the second row to the right as before. Minors are preserved and if the new matrix is positive definite so was the previous matrix. Continue this until we get a diagonal matrix with exactly the same (positive) minors as the original. Let's call the diagonal entries of this final matrix a_k. Then the quadratic form for this new matrix is $Q(X) = a_1x_1^2 + a_2x_2^2 + \ldots a_nx_n^2$. The positivity of the principal minors implies $a_k > 0$ for all k. This proves that this new quadratic form is positive definite and hence so is the original matrix A.

Next assume A is positive definite. It's easy to see that any proper special submatrix is positive definite. By induction its determinant is positive. In particular $a_{11} > 0$. By the previous argument we can preserve the determinant of A and positive definiteness by zeroing out entries in the first column and row using a_{11}. The determinant of the new matrix is the product of a_{11} and a special subminor. By induction this is positive. □